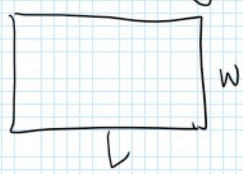
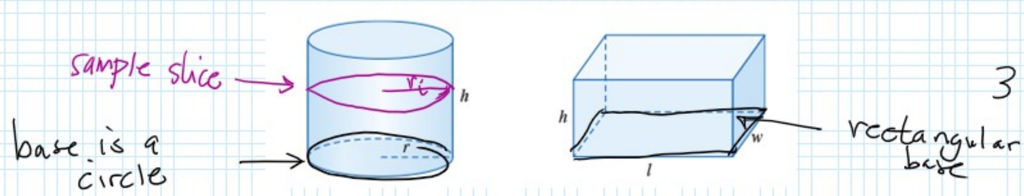
Section 6.2 - Volume

$A_{circle} = \pi r^2$

$A_{rectangle} = lw$



2 dimensional shapes



3 dimensional shape

cylinder

$V = A_{base} \cdot h$

$V_{CYL} = \pi r^2 \cdot h$

rectangular prism

$V = A_{base} \cdot h$

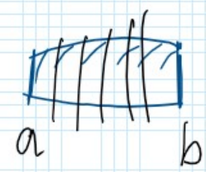
$V_{RP} = lwh$

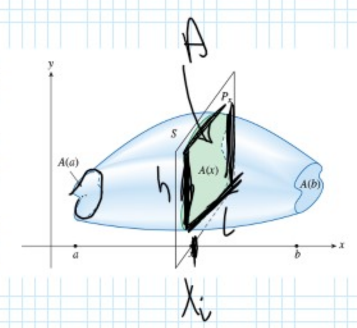
$A_{SLICES} \cdot h$

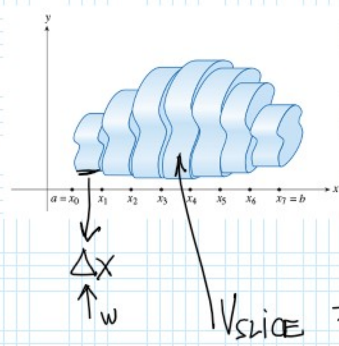
$\sum_{i=1}^{n} (\pi r_i)^2 \cdot h$

Volume of Uncommon Shape

ex. loaf of bread







function: $A(x)$

$V_{SLICE} = A \cdot \Delta x$

$\therefore V_{LOAF} \sum A_i \cdot \Delta x$

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x$$

$$\boxed{V = \int_a^b A(x)\,dx}$$

ex. use integration to show volume of a sphere with radius, r, is $\frac{4}{3}\pi r^3$

$$= \int_{-r}^{r} A_{SLICE}\,dx$$

$$= \int_{-r}^{r} \pi(r^2 - x^2)\,dx$$

$$V = 2\pi \int_0^r (r^2 - x^2)\,dx$$

$$= 2\pi \left(r^2 x - \frac{x^3}{3}\right)\Big|_0^r$$

$$= 2\pi \left(\frac{3r^3}{3} - \frac{1}{3}r^3\right)$$

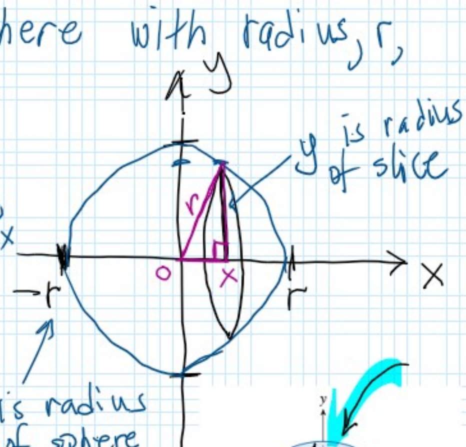
$$= 2\pi \, \frac{2r^3}{3}$$

$$V = \frac{4\pi r^3}{3}$$

$A_{SLICE} = \pi y^2$

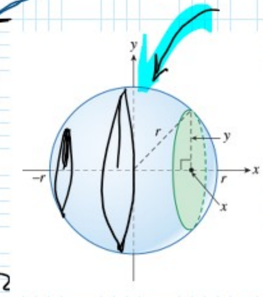
rewrite INTO x

$A(x) = \pi(r^2 - x^2)$



$x^2 + y^2 = r^2$

$y^2 = r^2 - x^2$



$= 2\pi \frac{2r^3}{3}$

$V_{SPHERE} = \frac{4\pi r^3}{3}$

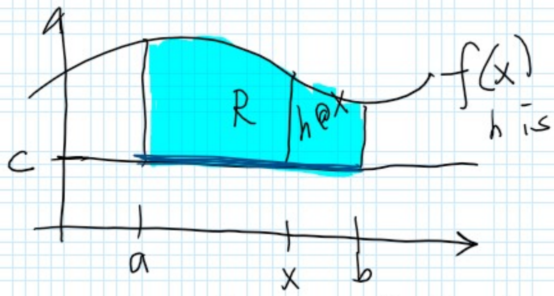
$x^2 + y^2 = r^2$

$y^2 = r^2 - x^2$

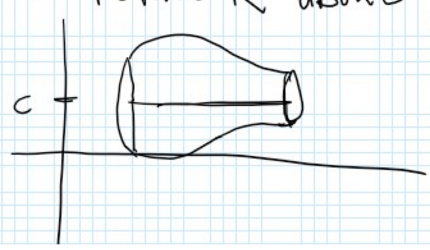
Create a Solid Using Revolution of 2-dim region

region, R, under curve $f(x)$ and above line $y = c$

revolve R about $y = c$

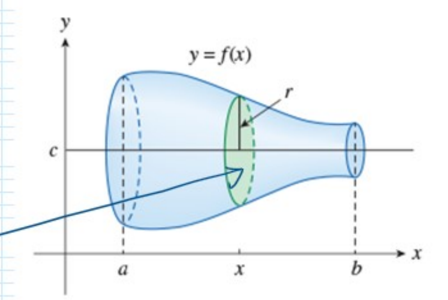


h is $f(x) - c$



Q: what is height of R at a given value of x?

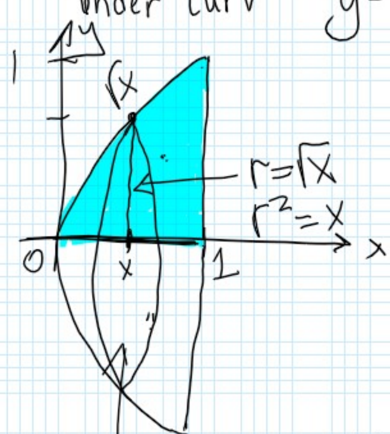
disk



$A_{slice} = \pi (f(x) - c)^2$

Find Volume Using Disk Method

ex. find volume of the solid obtained by rotating the region under curve $y = \sqrt{x}$ from $x = 0$ to $x = 1$ about the x-axis.



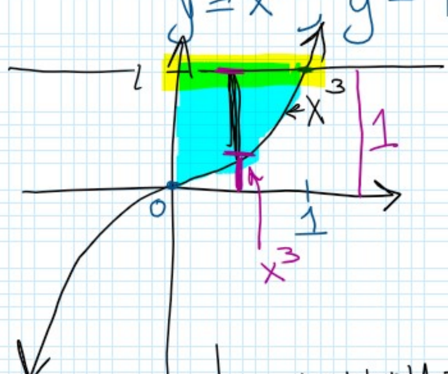
$A_{SLICE} = \pi r^2 = \pi x$

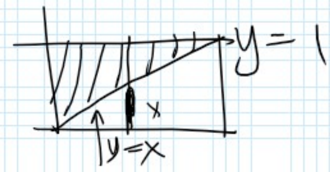
$= \pi \int_0^1 x \, dx$

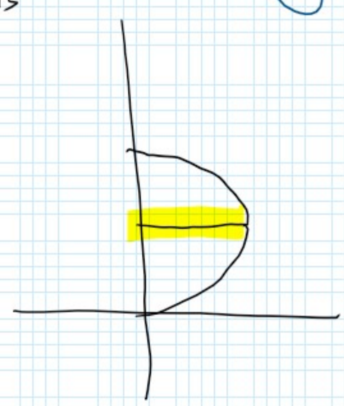
$= \pi \left. \frac{x^2}{2} \right|_0^1$

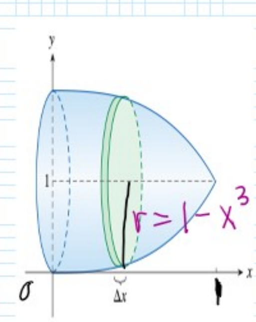
$= \frac{\pi}{2}(1) = \boxed{\frac{\pi}{2}}$

ex. find volume of solid obtained by rotating region bound by $y = x^3$, $y = 1$, $x = 0$ (y-axis) after being rotated about line $y = 1$.









$A_{SLICE} = \pi r^2$

$= \pi (1 - x^3)^2$

$= \pi (1 - 2x^3 + x^6)$

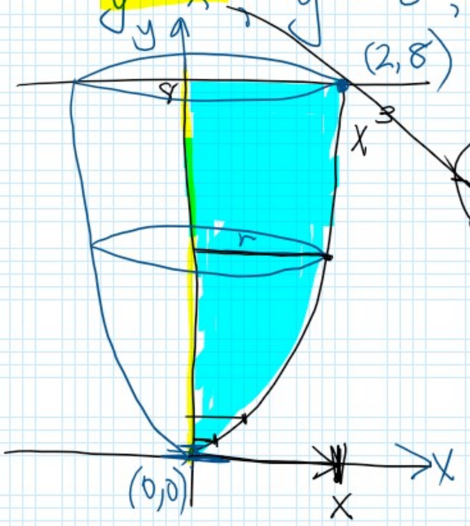
$V = \pi \int_0^1 (1 - 2x^3 + x^6) \, dx$

$= \pi \left(x - \frac{x^4}{2} + \frac{x^7}{7} \right) \Big|_0^1$

$= \pi \left(1 - \frac{1}{2} + \frac{1}{7} \right)$

$$= \pi \left(x - \frac{x^4}{2} + \frac{x^7}{7}\right)\Big|_0$$
$$= \pi \left(1 - \frac{1}{2} + \frac{1}{7}\right)$$
$$= \pi \left(\frac{14}{14} - \frac{7}{14} + \frac{2}{14}\right) = \boxed{\frac{9\pi}{14}}$$

ex. find volume of the solid obtained by rotating region bound by $y = x^3$, $y = 8$, $x = 0$ after being rotated about y-axis.



integrate WRT y

$y = x^3 \Rightarrow x = \sqrt[3]{y} = y^{1/3}$ ($x = r$)

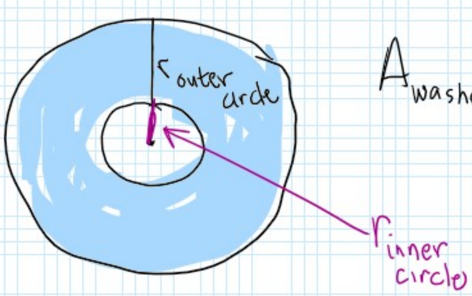
$r^2 = (y^{1/3})^2 = y^{2/3}$

$\int_0^8 A_{SLICE}\, dy$ (πr^2)

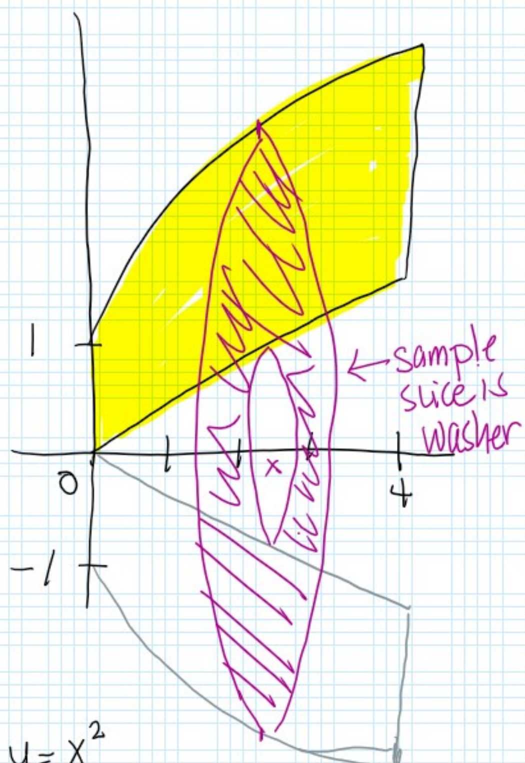
$$V = \pi \int_0^8 y^{2/3}\, dy$$
$$= \pi \frac{3}{5} y^{5/3} \Big|_0^8$$
$$= \frac{3\pi}{5} \cdot (8^{1/3})^5$$
$$2^5$$
$$\frac{3\pi}{5}\, 32 = \boxed{\frac{96\pi}{5}}$$

The Washer Method

explore region enclosed by curves $y = \sqrt{x} + 1$, $y = \frac{1}{2}x$ after rotating about x-axis on $[0,4]$



$$A_{washer} = A_{OUTER} - A_{INNER}$$



$$V_{SHAPE} = \int_a^b \left(\pi (r_{OUTER})^2 - \pi (r_{INNER})^2\right) dx$$

$$\boxed{V = \pi \int_a^b \left[(r_{OUTER})^2 - (r_{inner})^2\right] dx}$$

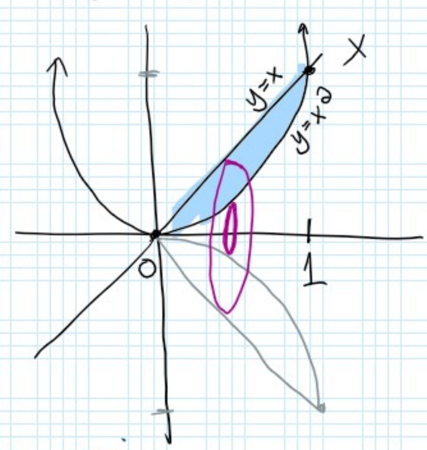
WASHER METHOD

ex. find the region enclosed by curves $y = x$, $y = x^2$ and is rotated about x-axis then find its volume

$$V = \pi \int_0^1 \left[x^2 - (x^2)^2\right] dx$$
$$= \pi \int_0^1 (x^2 - x^4)\, dx$$
$$= \pi \left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1$$
$$= \pi \left(\frac{1}{3} \cdot \frac{5}{5} - \frac{1}{5} \cdot \frac{3}{3}\right)$$
$$= \boxed{\frac{2\pi}{15}}$$

$x = x^2$
$0 = x^2 - x$
$x(x-1)$
$x = 0,\ x = 1$

$r_{OUTER} = x$
$r_{inner} = x^2$

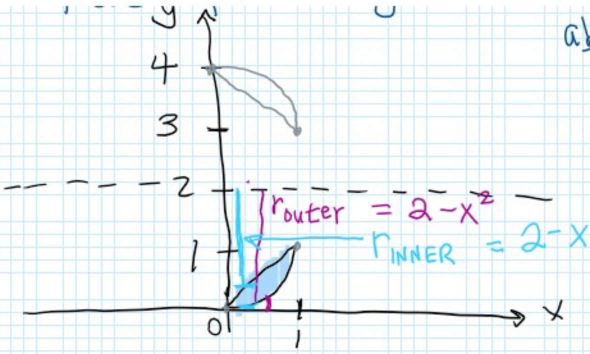


Food for Thought: what if rotated above region about line $y = 2$?

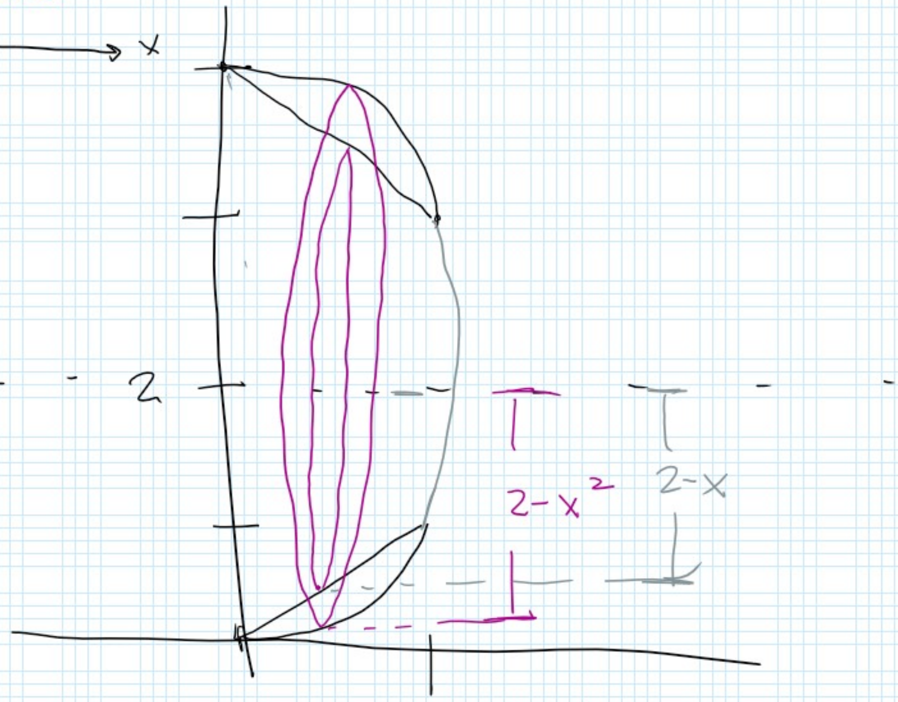
- sketch
- r_{OUTER}?

about line $y=2$?

- r_{OUTER} ?
- r_{INNER} ?



$$V = \pi \int_0^1 \left[(2-x^2)^2 - (2-x)^2\right] dx$$

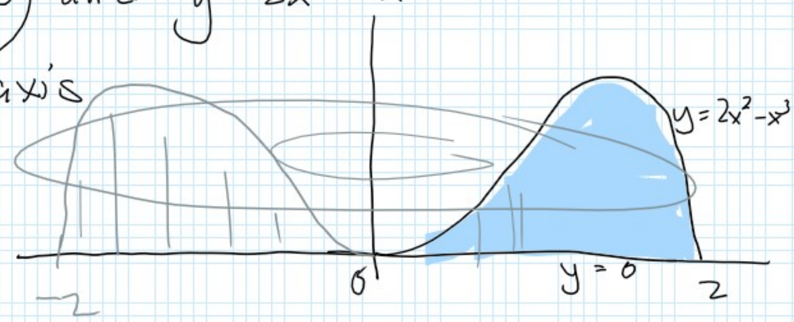


§6.3

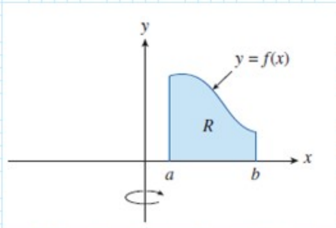
Sometimes volumes are impractical (if not impossible) to determine using disk or washer method

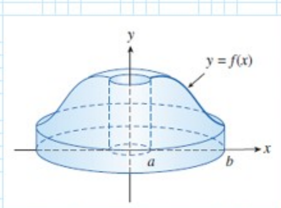
ex. region bound by $y=0$ and $y = 2x^2 - x^3$

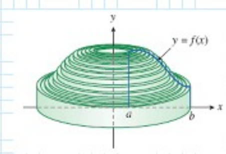
rotate about y-axis

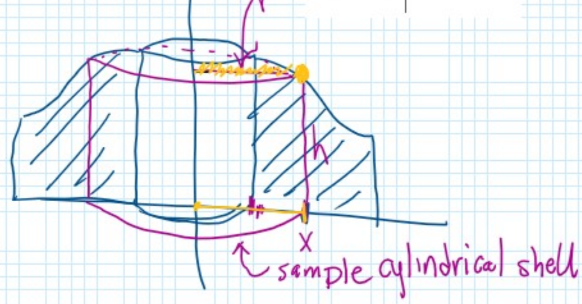


Alternate Method:
use cylindrical shells









region R is rotated about y-axis

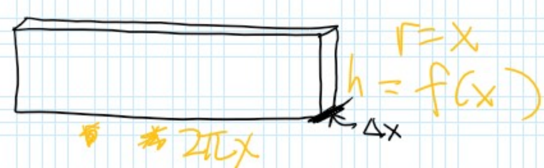
where $0 \le a < b$

$$\boxed{V = 2\pi \int_a^b \underset{(r)}{x}\, \underset{(h)}{f(x)}\, dx}$$

Volume using cylindrical shells

$V_{CYL} = \pi r^2 h$

shell opened is a rectangular prism



ex. find volume of solid obtained by region bound by $y=0$ and $y=2x^2-x^3$ and rotated about y-axis

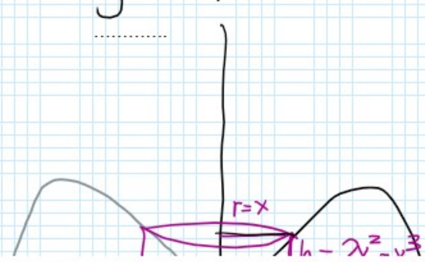
find bounds: $2x^2 - x^3 = 0$

$x^2(2-x) = 0$

$x=0 \quad x=2$

$$V = 2\pi \int_0^2 x(2x^2 - x^3)\, dx$$

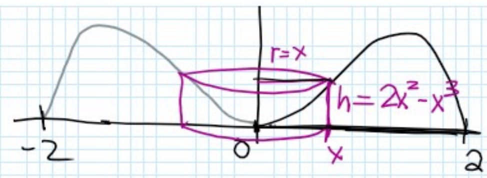
$$= 2\pi \int_0^2 (2x^3 - x^4)\, dx$$



$= 2\pi \int_0^2 (2x^3 - x^4)\,dx$

$= 2\pi \left(\frac{x^4}{2} - \frac{x^5}{5}\right)\Big|_0^2$ ü

$= 2\pi \left(8\cdot\frac{5}{5} - \frac{32}{5}\right) = 2\pi \cdot \frac{8}{5} = \boxed{\frac{16\pi}{5}}$ ü



ex. find volume of solid obtained by rotating about y-axis the region between $y = x$ and $y = x^2$

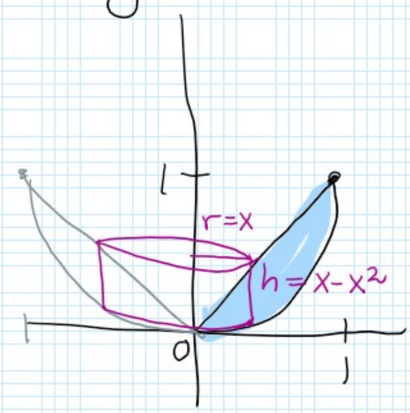
$V = 2\pi \int_0^1 \underbrace{x}_{r}\,\underbrace{(x - x^2)}_{h}\,dx$

$= 2\pi \int_0^1 (x^2 - x^3)\,dx$

$= 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1$

$= 2\pi \left(\frac{1}{3}\cdot\frac{4}{4} - \frac{1}{4}\cdot\frac{3}{3}\right)$

$= \cancel{2}\pi \left(\frac{4-3}{\cancel{12}_{6}}\right) = \boxed{\frac{\pi}{6}}$



region bound by $y = x - x^2$ and $y = 0$
rotated about $x = 2$

$y = -x^2 + x$

x-int : $x = 0, 1$
y-int : $x = 0$

$x = \frac{-1}{2(-1)} = \frac{1}{2}$

$f\left(\frac{1}{2}\right) = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$

vertex : $\left(\frac{1}{2}, \frac{1}{4}\right)$

$V_{\text{SOLID}} = 2\pi \int_0^1 \overset{r}{(2-x)}\,\overset{h}{(x - x^2)}\,dx = \boxed{\frac{\pi}{2}}$